Ingegneria delle Telecomunicazioni

Satellite Communications

17. GNSS, does it work? Even with smartphones?

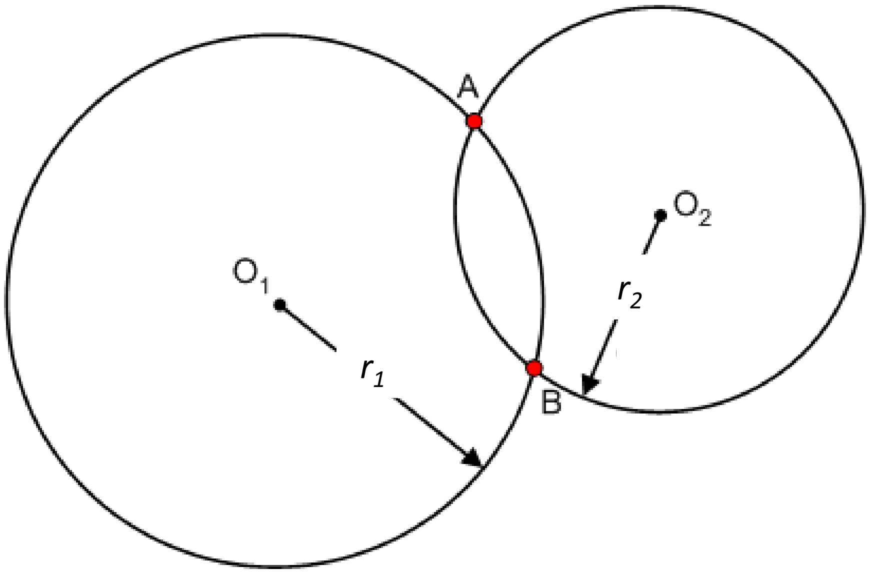
Marco Luise

marco.luise@unipi.it

Dip. Ingegneria dell'Informazione, Univ. Pisa, Italy

Main Problem: 2D Positioning from range measurements

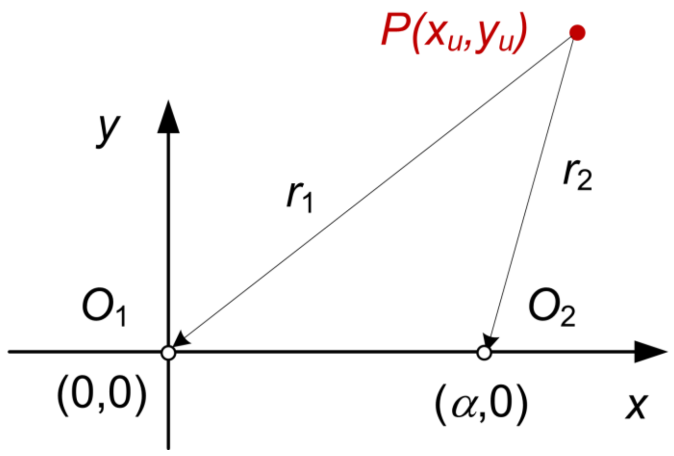
*"The position of a certain point in space can be found from distances (**ranges**) measured from this point to some other known positions in space"*



- O_1 and O_2 represent the Satellites of a GNSS system
- The receiver owned by Alice is at the point A
- The range *r* can be derived from a propagation-time (flight time, travel time) measurement τ,
- $r = c \cdot \tau$ (*c*=speed of light)

Ambiguity: Both A and B are solutions of the problem!

Example

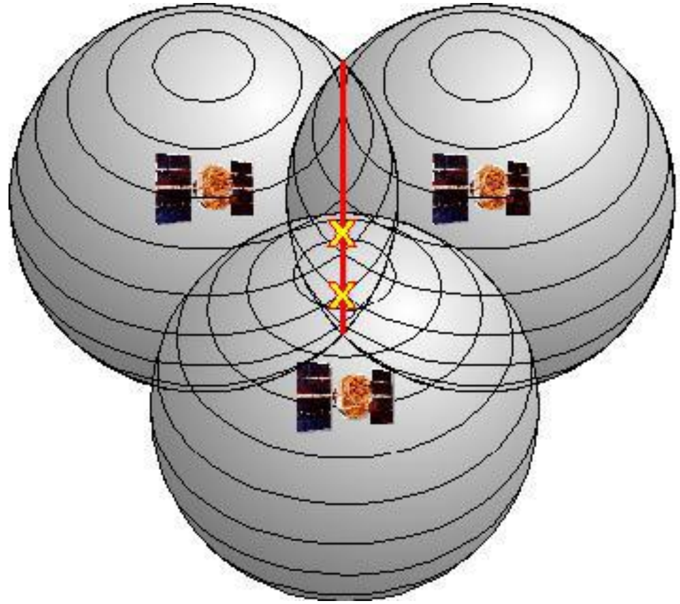


Range Measurement & Positioning

$\mathbf{r}_i=(x_i,y_i,z_i)$ ECEF coordinates of satellite #*i* (known)

$\mathbf{r}=(x_u,y_u,z_u)$ ECEF coordinate of the receiver (unknown)

r_i=range from satellite #*i* to receiver (measured)



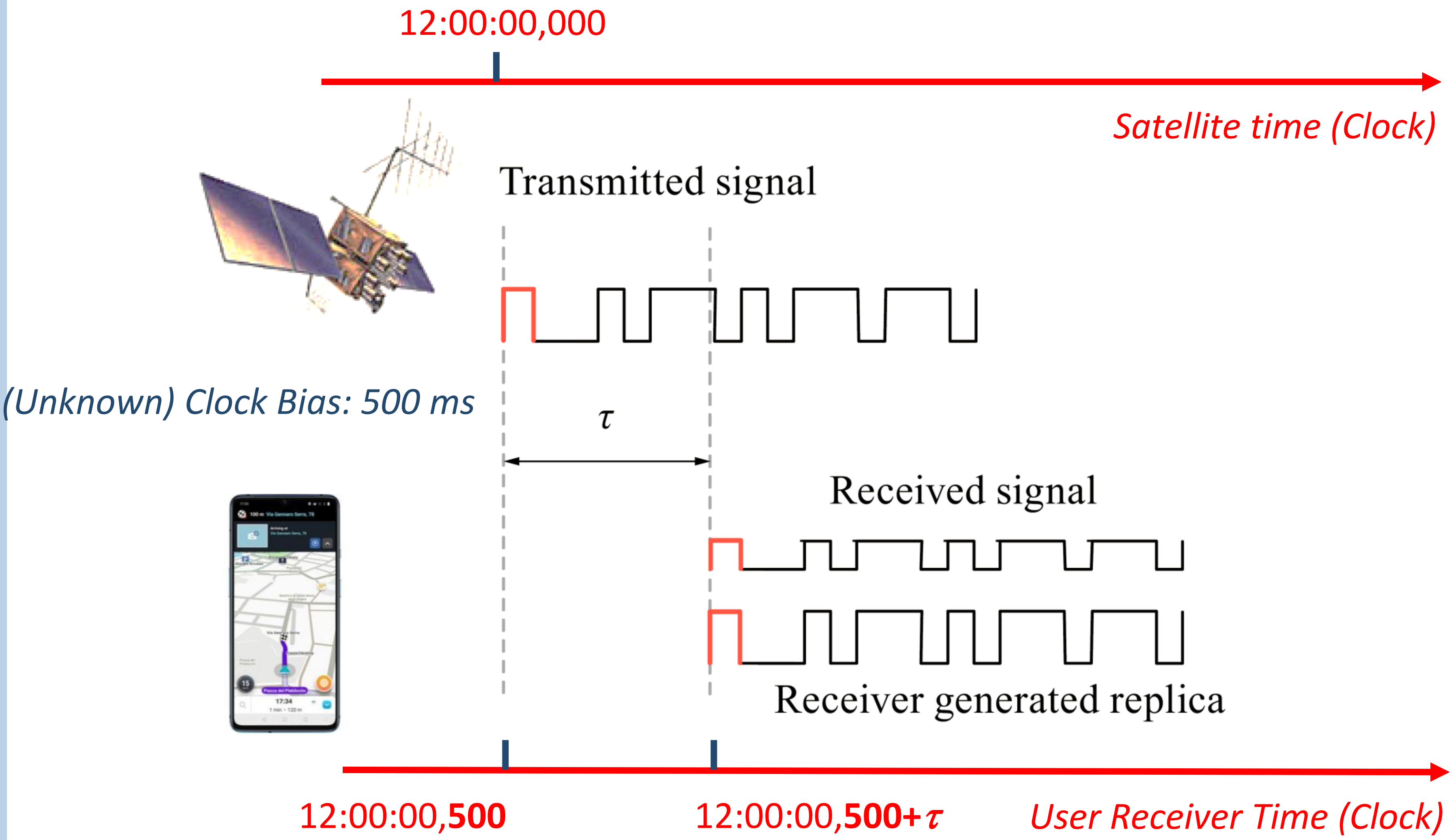
$$\begin{cases} \sqrt{(x_u-x_1)^2+(y_u-y_1)^2+(z_u-z_1)^2}=r_1 \\ \sqrt{(x_u-x_2)^2+(y_u-y_2)^2+(z_u-z_2)^2}=r_2 \\ \sqrt{(x_u-x_3)^2+(y_u-y_3)^2+(z_u-z_3)^2}=r_3 \end{cases}$$

Three unknowns, three (independent) equations

DONE!

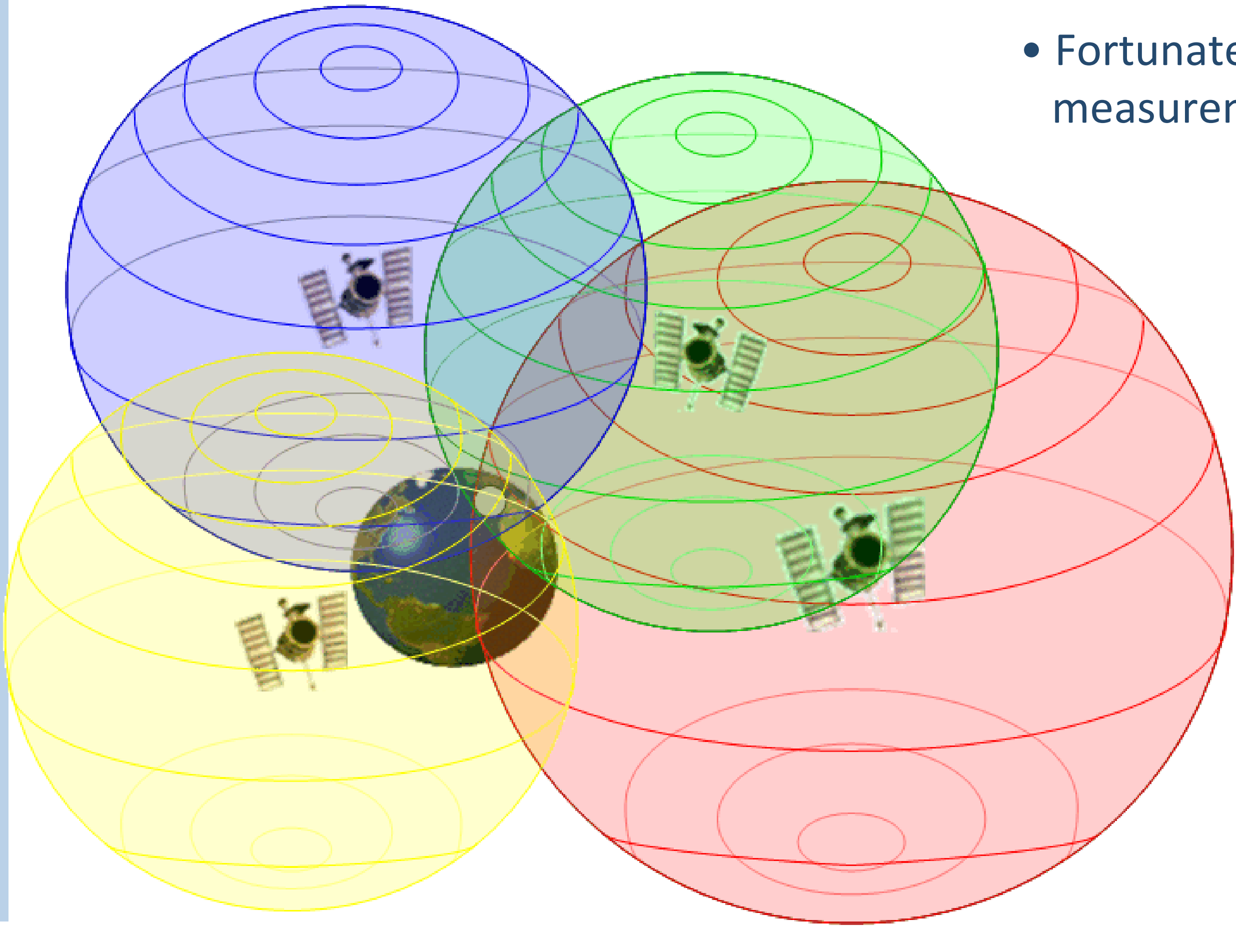
You try, and you find you're in outer space.

Measuring range: the clock bias



Range and Pseudrange

- What is actually measured is a pseudorange ρ, containing the *unknown* clock bias effect



- Fortunately, the bias is the same for all measurement – can be considered as a fourth unknown to be found
- WE NEED ONE MORE SATELLITE/ OBSERVATION/ EQUATION
- Minimum # of satellites in view (received) for GNNS to work: **4**

The (Nonlinear) Positioning Equations

$$
\begin{cases}
\sqrt{(x_u - x_1)^2 + (y_u - y_1)^2 + (z_u - z_1)^2} + c\Delta t = \rho_1 \\
\sqrt{(x_u - x_2)^2 + (y_u - y_2)^2 + (z_u - z_2)^2} + c\Delta t = \rho_2 \\
\sqrt{(x_u - x_3)^2 + (y_u - y_3)^2 + (z_u - z_3)^2} + c\Delta t = \rho_3 \\
\sqrt{(x_u - x_4)^2 + (y_u - y_4)^2 + (z_u - z_4)^2} + c\Delta t = \rho_4
\end{cases}
$$

$$\boldsymbol{\rho} \triangleq (\rho_1, \rho_2, \rho_3, \rho_4)^T, \quad \boldsymbol{\xi} \triangleq (x_u, y_u, z_u, c \cdot \Delta t)^T$$

$$f_i(\boldsymbol{\xi}) \triangleq \sqrt{(x_u - x_i)^2 + (y_u - y_i)^2 + (z_u - z_i)^2} + c \cdot \Delta t$$

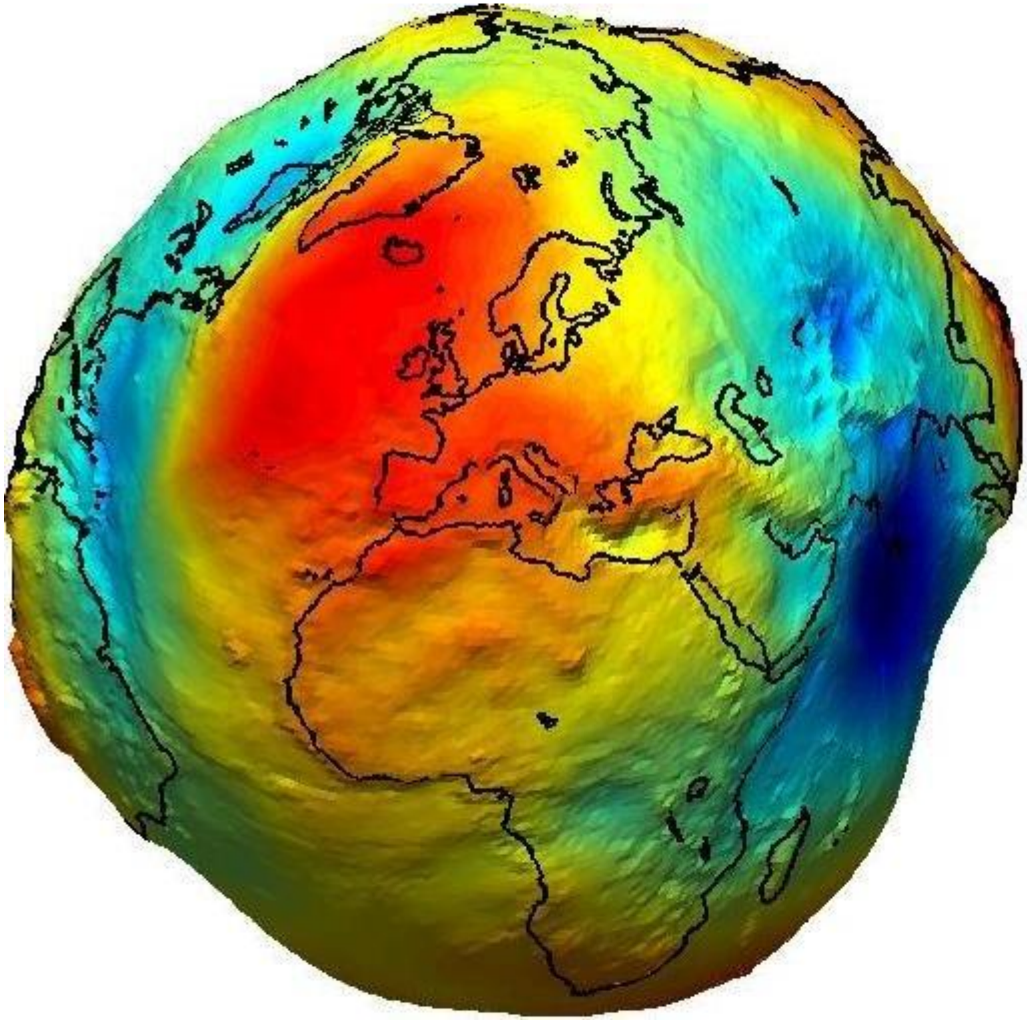
$$\mathbf{f}(\boldsymbol{\xi}) \triangleq \left(f_1(\boldsymbol{\xi}), f_2(\boldsymbol{\xi}), f_3(\boldsymbol{\xi}), f_4(\boldsymbol{\xi}) \right)^T$$

$$\mathbf{f}(\boldsymbol{\xi}) = \boldsymbol{\rho}$$

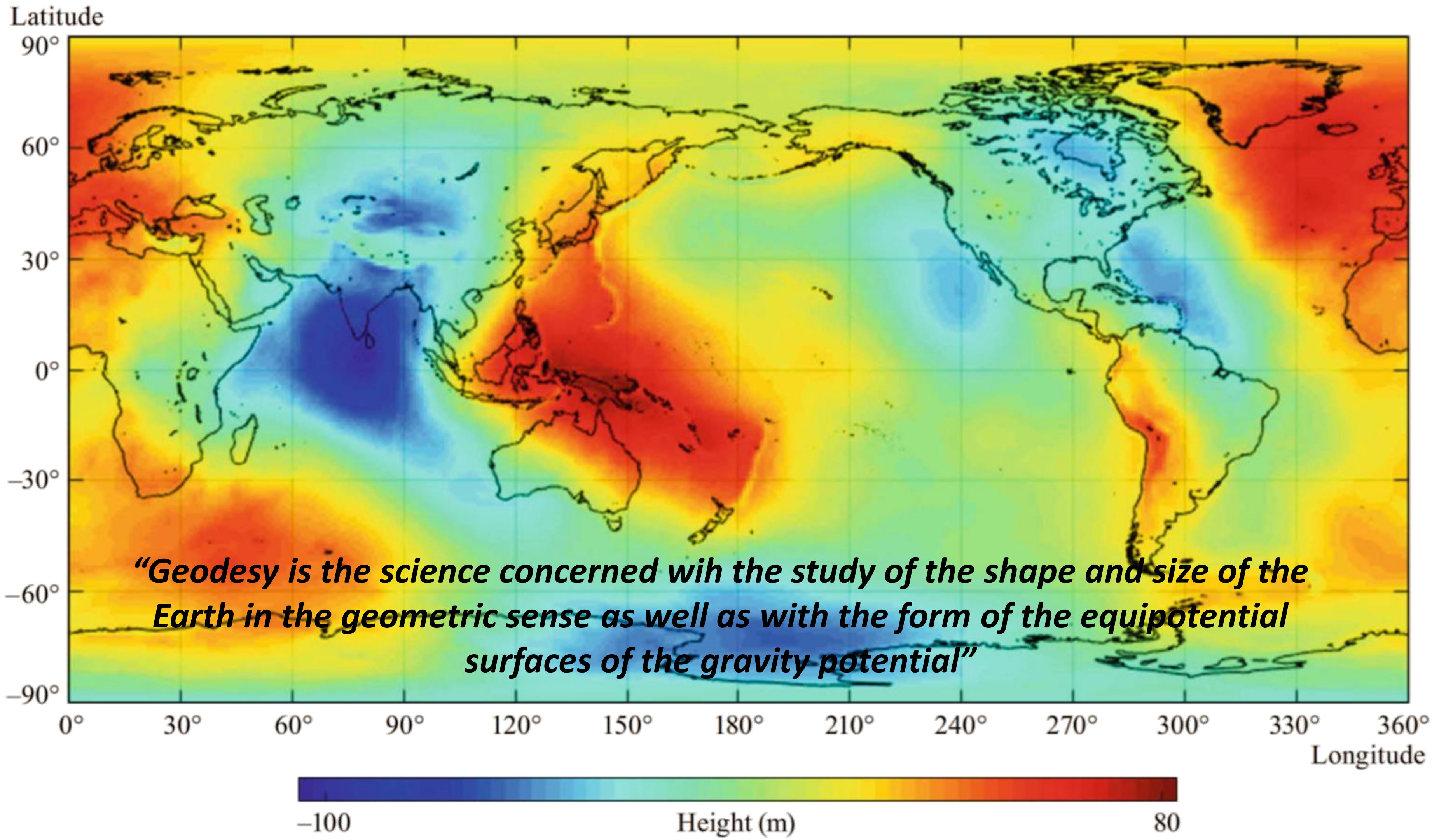
The shape of the Earth

- **The Earth is Flat**
  - In early Egyptian and Mesopotamian thought, the world was portrayed as a disk floating in the ocean
- **The Earth is a Sphere**
  - The founder of scientific geodesy was Eratosthenes (276-195 BC) of Alexandria who, assuming the Earth was spherical, deduced from measurements a radius for the Earth.
- **The Earth is an Ellipsoid**
  - Towards the end of the 17th century, Newton demonstrated that the concept of a truly spherical Earth was inadequate as an explanation of the equilibrium of the ocean surface, owing to the Earth rotation: he showed, by means of a simple theoretical model, that the hydrostatic equilibrium would be maintained if the equatorial axis were longer than the polar axis. This is equivalent to the statement that the body is flattened towards the pole.
- **The Earth is the *Geoid***
  - Listing (1873) had given the name **geoid** to the “equipotential surface of the Earth’s gravity field which would coincide with the ocean surface, if the Earth were undisturbed and without topography”.

The (exaggerated) Geoid

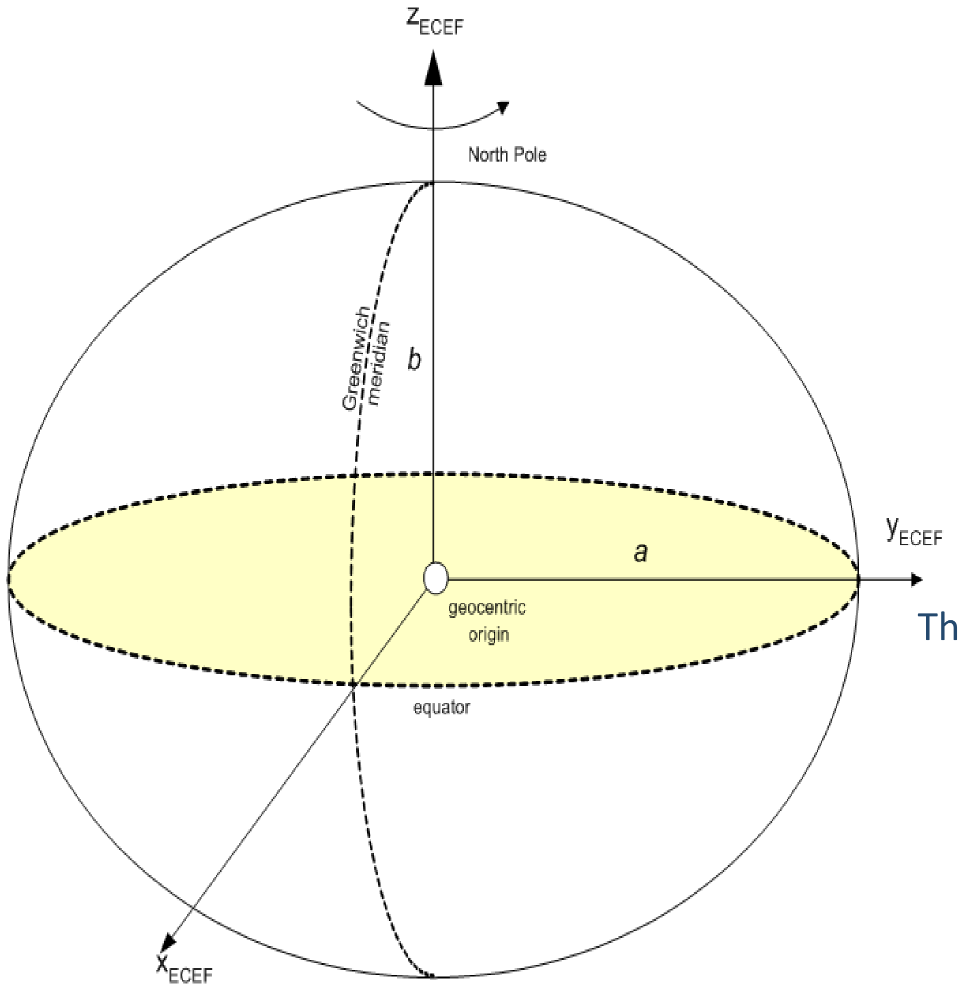


Difference between the Geoid and the (average) Ellipsoid



"Geodesy is the science concerned wih the study of the shape and size of the Earth in the geometric sense as well as with the form of the equipotential surfaces of the gravity potential"

(Cartesian) Earth-Centered Earth-Fixed (ECEF) coordinates



Origin: Earth's center of mass

Z-Axis: direction of mean rotational axis of Earth

X-Axis: intersection of Greenwich meridian and the plane passing through the origin and normal to the Z-Axis

Y-Axis: direction orthogonal to Z-Axis and X-Axis

The reference system for our positioning equations

$$
\begin{cases}
\sqrt{(x_u - x_1)^2 + (y_u - y_1)^2 + (z_u - z_1)^2} + c\Delta t = \rho_1 \\
\sqrt{(x_u - x_2)^2 + (y_u - y_2)^2 + (z_u - z_2)^2} + c\Delta t = \rho_2 \\
\sqrt{(x_u - x_3)^2 + (y_u - y_3)^2 + (z_u - z_3)^2} + c\Delta t = \rho_3 \\
\sqrt{(x_u - x_4)^2 + (y_u - y_4)^2 + (z_u - z_4)^2} + c\Delta t = \rho_4
\end{cases}
$$

Geocentric ECEF coordinates

The Earth is assumed to be spherical

distance:

$$r = \sqrt{x_{ECEF}^2 + y_{ECEF}^2 + z_{ECEF}^2}$$

latitude:

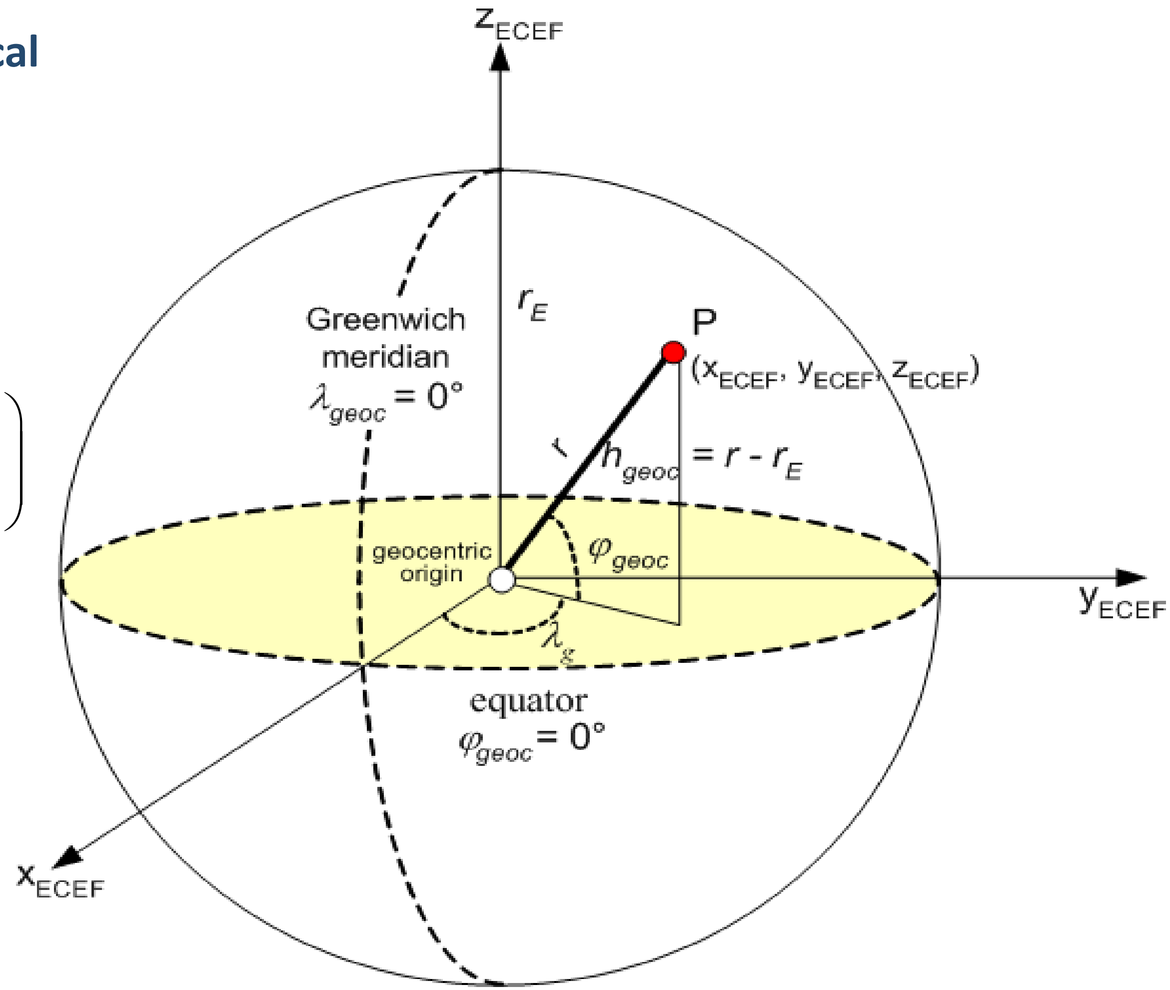
$$\varphi_{geoc} = \arctan\left(\frac{z_{ECEF}}{\sqrt{x_{ECEF}^2 + y_{ECEF}^2}}\right)$$

longitude:

$$\lambda_{geoc} = \arctan\left(\frac{y_{ECEF}}{x_{ECEF}}\right)$$

altitude:

$$h_{geoc} = r - r_E$$



The World Geodetic System – 1984 (WGS-84)

The most used and very accurate reference frame is the **World Geodetic System–1984** *(WGS-84) Reference Ellipsoid*

Parameter	**Symbol**	**Value**
Semi-major axis	a	6378137 m
Eccentricity	$e_e = \sqrt{\frac{a^2 - b^2}{a^2}}$	0.0818191908426
Flattening	$e_p = \frac{a - b}{a}$	$\frac{1}{298.277223563}$

WGS-84 Reference Ellipsoid

Geodetic coordinates

Assuming the Earth as an ellipsoid:

latitude:

$$\varphi_{geod} = \arctan\left[\frac{z_{ECEF} + \dfrac{2e_p - e_p^2}{1-e_p}\cdot a\cdot\sin^3\theta}{p-\left(2e_p - e_p^2\right)\cdot a\cdot\cos^3\theta}\right]$$

longitude:

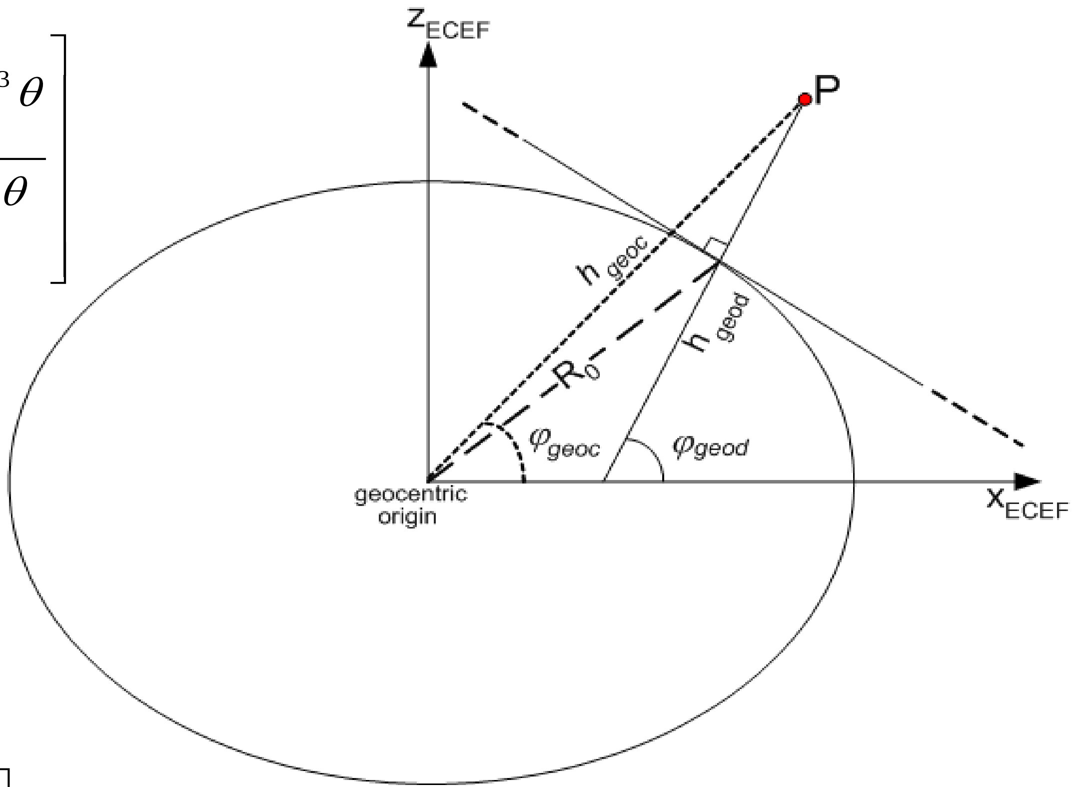
$$\lambda_{geod} = \lambda_{geoc} = \arctan\left(\frac{y_{ECEF}}{x_{ECEF}}\right)$$

altitude:

$$h_{geod} = \frac{p}{\cos\varphi_{geod}} - \nu$$

where:

$$p = \sqrt{x_{ECEF}^2 + y_{ECEF}^2}, \quad \theta = \arctan\left[\frac{z_{ECEF}}{p\cdot(1-e_p)}\right], \quad \nu = \frac{a}{\sqrt{1-e_e^2\cdot\sin^2\varphi_{geod}}}$$



Topocentric reference system (1/2)

User-centric reference that we use to locate a Satellte in the sky or a celestial body with respect to the observer position

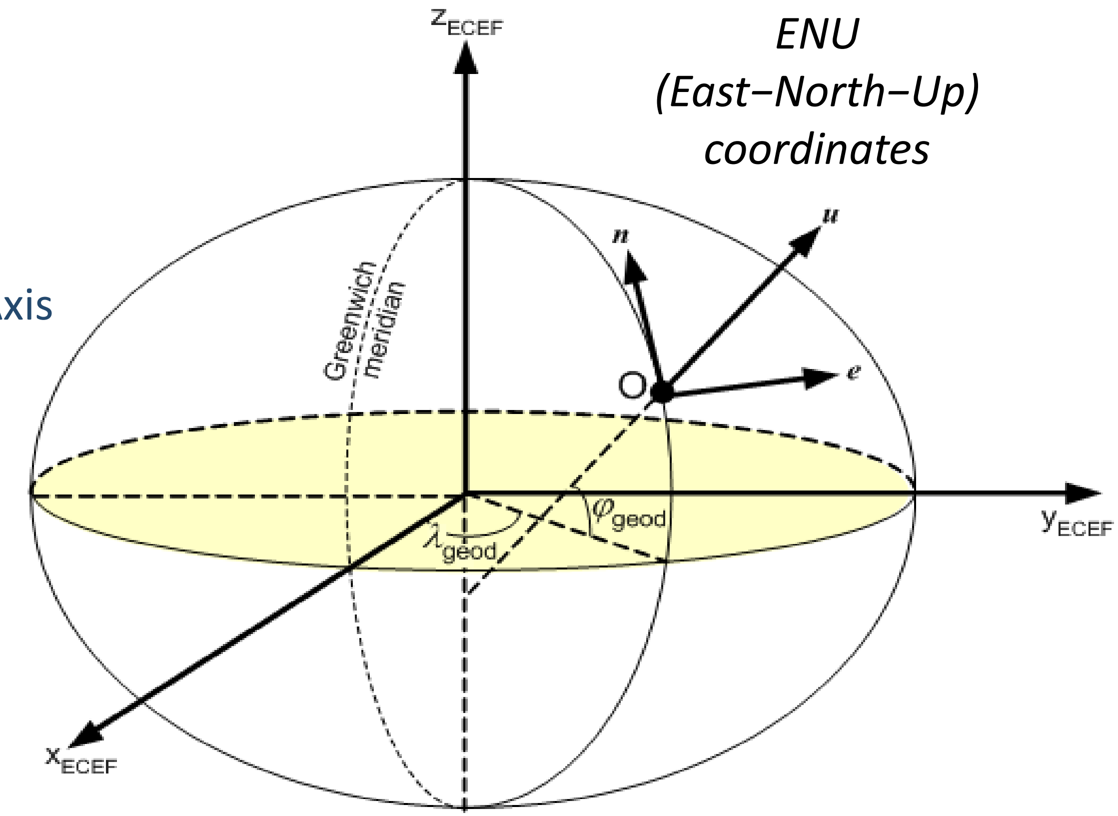
Origin: observer's position

u-Axis: direction of local vertical

n-Axis: direction of the North pole

e-Axis: direction orthogonal to u-Axis and n-Axis

ENU (East−North−Up) coordinates



Topocentric reference system (2/2)

$$\Delta\xi = \{\Delta x, \Delta y, \Delta z\} = \xi_C - \xi_O, \quad \xi = \{x_{ECEF}, y_{ECEF}, z_{ECEF}\}$$

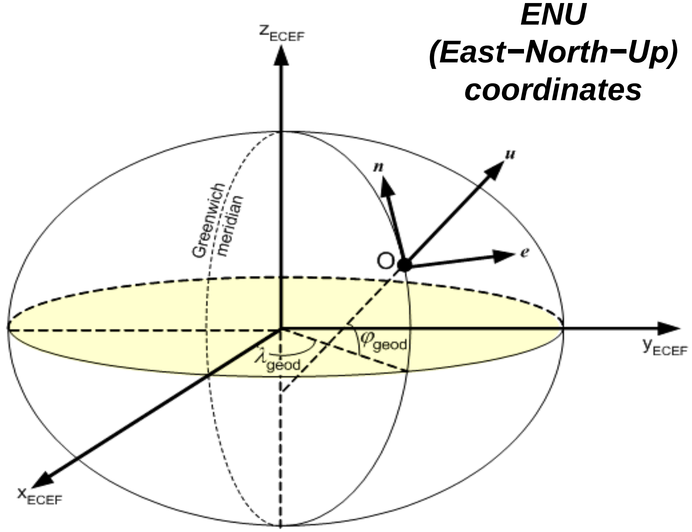
$$\begin{pmatrix} E \\ N \\ U \end{pmatrix} = \begin{pmatrix} -\sin\lambda_{geod} & \cos\lambda_{geod} & 0 \\ -\sin\varphi_{geod}\cos\lambda_{geod} & -\sin\varphi_{geod}\sin\lambda_{geod} & \cos\varphi_{geod} \\ \cos\varphi_{geod}\cos\lambda_{geod} & \cos\varphi_{geod}\sin\lambda_{geod} & \sin\varphi_{geod} \end{pmatrix} \cdot \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

***elevation*:** $\varepsilon = \arctan\left(U/\sqrt{E^2 + N^2}\right)$

***azimuth*:** $\alpha = \arctan(E/N)$

***range*:** $\rho = \sqrt{E^2 + N^2 + U^2}$

ENU (East−North−Up) coordinates



Back to the (Nonlinear) Positioning Equations

$$\begin{cases} \sqrt{(x_u - x_1)^2 + (y_u - y_1)^2 + (z_u - z_1)^2} + c\Delta t = \rho_1 \\ \sqrt{(x_u - x_2)^2 + (y_u - y_2)^2 + (z_u - z_2)^2} + c\Delta t = \rho_2 \\ \sqrt{(x_u - x_3)^2 + (y_u - y_3)^2 + (z_u - z_3)^2} + c\Delta t = \rho_3 \\ \sqrt{(x_u - x_4)^2 + (y_u - y_4)^2 + (z_u - z_4)^2} + c\Delta t = \rho_4 \end{cases}$$

$$\boldsymbol{\rho} \triangleq (\rho_1, \rho_2, \rho_3, \rho_4)^T, \quad \boldsymbol{\xi} \triangleq (x_u, y_u, z_u, c \cdot \Delta t)^T$$

$$f_i(\boldsymbol{\xi}) \triangleq \sqrt{(x_u - x_i)^2 + (y_u - y_i)^2 + (z_u - z_i)^2} + c \cdot \Delta t$$

$$\mathbf{f}(\boldsymbol{\xi}) \triangleq (f_1(\boldsymbol{\xi}), f_2(\boldsymbol{\xi}), f_3(\boldsymbol{\xi}), f_4(\boldsymbol{\xi}))^T$$

$$\mathbf{f}(\boldsymbol{\xi}) = \boldsymbol{\rho}$$

Dip. Ingegneria dell'Informazione
University of Pisa, Italy

Marco Luise

Iterative Solution (Linearization)

- **Nonlinear Positioning Equation:** $\mathbf{f}(\boldsymbol{\xi}) - \boldsymbol{\rho} = \mathbf{0}$
- **Iterative solution (Newton):** $\boldsymbol{\xi}^{(n+1)} = \boldsymbol{\xi}^{(n)} - \left(\mathbf{Jf}(\boldsymbol{\xi}^{(n)})\right)^{-1}\left(\mathbf{f}(\boldsymbol{\xi}^{(n)}) - \boldsymbol{\rho}\right)$

$$\mathbf{Jf}(\boldsymbol{\xi}) = \begin{pmatrix} \frac{x_u - x_1}{r_1} & \frac{y_u - y_1}{r_1} & \frac{z_u - z_1}{r_1} & 1 \\ \frac{x_u - x_2}{r_2} & \frac{y_u - y_2}{r_2} & \frac{z_u - z_2}{r_2} & 1 \\ \frac{x_u - x_3}{r_3} & \frac{y_u - y_3}{r_3} & \frac{z_u - z_3}{r_3} & 1 \\ \frac{x_u - x_4}{r_4} & \frac{y_u - y_4}{r_4} & \frac{z_u - z_4}{r_4} & 1 \end{pmatrix}$$

$$r_i \triangleq \sqrt{(x_u - x_i)^2 + (y_u - y_i)^2 + (z_u - z_i)^2}$$

- **The Jacobian matrix is updated at any iteration**
- **We need to accurately know the positions of the *anchors* (the satellites in the sky)**
- **GNSS receivers typically provide solutions every second – there's a time of 1s to perfect iterations**

Computing the user *velocity* – Doppler shift measurement

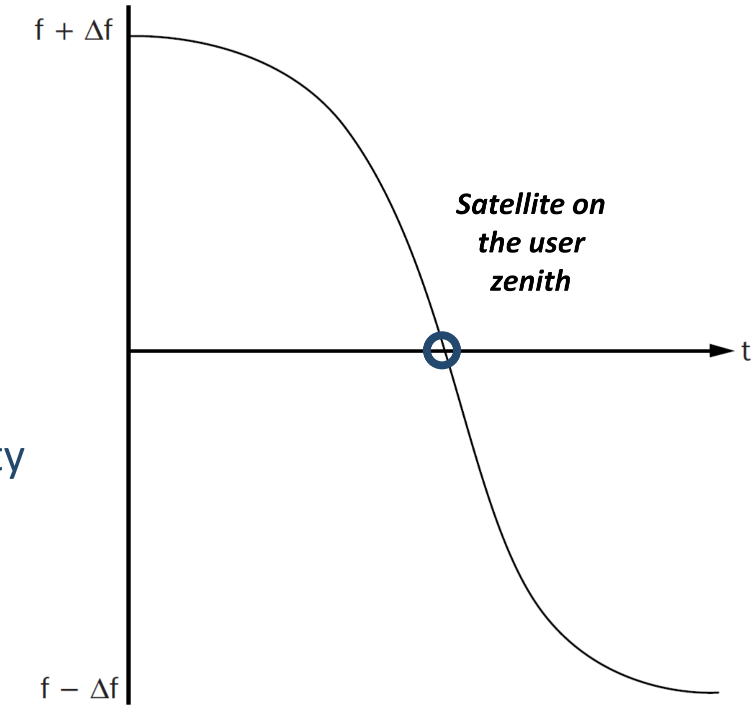
- It is usually derived from observation of *carrier frequency Doppler shift* Δf_i on satellite # *i* – deriving velocity as $\dot{\xi}$ is no good because of error propagation

$$f_i = \left(1 - \frac{\mathbf{r}_i}{\|\mathbf{r}_i\|} \cdot \frac{\mathbf{v}_i}{c}\right) f_c = f_c + \Delta f_i$$

$$\mathbf{r}_i = \left(x_u - x_i, y_u - y_i, z_u - z_i\right)^T$$

- What matters is the *radial* component of the satellite velocity wrt the user
- If the user receiver is *moving itself* with velocity $\mathbf{v}_u$ (the unknown that we wish to estimate)

$$f_i = \left(1 - \frac{\mathbf{r}_i}{\|\mathbf{r}_i\|} \cdot \frac{(\mathbf{v}_i - \mathbf{v}_u)}{c}\right) f_c$$



Computing velocity and clock drift

- Assumption: the satellites speeds are *known* because they can be derived from the navigation message (ephemerides)
- The carrier frequencies of the satellites *should* all be equal to the nominal system f_c, BUT they are different because the different atomic clocks onboard the satellites may by slightly different – the corrections are sent down in the navigation message, so the actual *individual* transmitted frequency $f_{c,i}$ by satellite *i* is known, and the individual *i*-th *received* frequency is
-

$$f_i = \left(1 - \frac{\mathbf{r}_i}{\left\|\mathbf{r}_i\right\|} \cdot \frac{(\mathbf{v}_i - \mathbf{v}_u)}{c}\right) f_{c,i}$$

- Further problem: the actual, *measured* received frequency $f_{R,i}$ is *inaccurate* as it contains an (unknown) *frequency bias* δ of the local oscillator (common to all observations): $f_{R,i} = (1+\delta) f_i$ and we have a further unknown (as in positioning…)

$$(1+\delta)\, f_i = \left(1 - \boldsymbol{\alpha}_i \cdot \frac{(\mathbf{v}_i - \mathbf{v}_u)}{c}\right) f_{c,i} \quad , \quad \boldsymbol{\alpha}_i \triangleq \frac{\mathbf{r}_i}{\left\|\mathbf{r}_i\right\|}$$

Velocity Equations (4 satellites)

$$\begin{cases} \dfrac{cf_1}{f_{0,1}} + \dfrac{cf_1}{f_{0,1}}\delta = c - \alpha_{x,1}(v_{x,1} - v_{x,u}) - \alpha_{y,1}(v_{y,1} - v_{y,u}) - \alpha_{z,1}(v_{z,1} - v_{z,u}) \\ \dfrac{cf_2}{f_{0,2}} + \dfrac{cf_2}{f_{0,2}}\delta = c - \alpha_{x,2}(v_{x,2} - v_{x,u}) - \alpha_{y,2}(v_{y,2} - v_{y,u}) - \alpha_{z,2}(v_{z,2} - v_{z,u}) \\ \dfrac{cf_3}{f_{0,3}} + \dfrac{cf_3}{f_{0,3}}\delta = c - \alpha_{x,3}(v_{x,3} - v_{x,u}) - \alpha_{y,3}(v_{y,3} - v_{y,u}) - \alpha_{z,3}(v_{z,3} - v_{z,u}) \\ \dfrac{cf_4}{f_{0,4}} + \dfrac{cf_4}{f_{0,4}}\delta = c - \alpha_{x,4}(v_{x,4} - v_{x,u}) - \alpha_{y,4}(v_{y,4} - v_{y,u}) - \alpha_{z,4}(v_{z,4} - v_{z,u}) \end{cases}$$

- The **unknowns** are $\boldsymbol{v_{x,u}}$, $\boldsymbol{v_{y,u}}$, $\boldsymbol{v_{z,u}}$, and the clock offset δ, all of the other quantities are known – the equations are *linear*
- As a side effect, the clock-rate bias δ is also derived
- Of course, previous derivation of position is needed to compute α

Measurement Errors & Noise Propagation

- Many error sources affect the measurement of the pseudo range(s) (receiver noise, ionosphere, troposphere, multipath, etc.).
- They can be collectively modeled as *an observation noise vector* **w** that adds up to $\boldsymbol{\rho}$, but cannot be of course separated from it during observation, therefore "propagate" down to the solution ξ.
- The total noise variance, i.e., the sum of the variance on the 4 (pseudo)range components (including time) induced by noise randomness is called the *User Equivalent Range Error* (UERE)
- It is fundamental to understand the *propagation* of such errors down to the (iterative) solution of the positioning equations

New Observation Model with Errors:

$$\mathbf{f}(\boldsymbol{\xi}) = \boldsymbol{\rho} + \mathbf{w}$$

Linear Analysis of Noise Propagation

- New Model: $\mathbf{f}(\boldsymbol{\xi}) = \boldsymbol{\rho} + \mathbf{w}$

- Linearized observation model around the true position ξ_u :

$$\mathbf{f}(\boldsymbol{\xi}_u) + \mathbf{A}(\boldsymbol{\xi} - \boldsymbol{\xi}_u) = \boldsymbol{\rho} + \mathbf{w} \quad , \quad \mathbf{A} \triangleq \mathbf{Jf}(\boldsymbol{\xi}_u)$$

- Or, introducing the perturbation on the solution caused by noise $\Delta\xi=\xi-\xi_u$ and observing that by definition $\mathbf{f}(\xi_u)=\rho$,

$$\mathbf{A}\Delta\boldsymbol{\xi} = \mathbf{w}$$

- so that the noise *propagation equation* is:

$$\Delta\boldsymbol{\xi} = \mathbf{A}^{-1}\mathbf{w}$$

Dilution of Precision

- We can find the *covariance matrix* $\mathbf{C}_\Delta$ of the propagated noise

$$\mathbf{C}_\Delta = E\left\{\Delta\boldsymbol{\xi}\Delta\boldsymbol{\xi}^T\right\} = E\left\{\mathbf{A}^{-1}\mathbf{w}\mathbf{w}^T\left(\mathbf{A}^T\right)^{-1}\right\} = \mathbf{A}^{-1}E\left\{\mathbf{w}\mathbf{w}^T\right\}\left(\mathbf{A}^T\right)^{-1}$$

- In a first approximation, we can assume that the components of **w** are zero-mean and *uncorrelated* (not completely true for instance for the *Iono* term), so that

$$\mathbf{C}_\Delta = \mathbf{A}^{-1}\mathbf{C}_w\left(\mathbf{A}^T\right)^{-1} = \mathbf{A}^{-1}\sigma_w^2\mathbf{I}\left(\mathbf{A}^T\right)^{-1} = \sigma_w^2\left(\mathbf{A}^T\mathbf{A}\right)^{-1}$$

- The equation describes the so-called *dilution of precision*, i.e, how the measurement inaccuracy propagates down to the positioning solution.
- An overall metrics of positioning error (including clock bias) is the total variance, that is, the sum of the four variances of the four positioning components:

$$\sigma_{tot}^2 \triangleq \sigma_{\xi_1}^2 + \sigma_{\xi_2}^2 + \sigma_{\xi_3}^2 + \sigma_{\xi_4}^2 = \sigma_w^2\,\mathrm{tr}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}$$

Different DOPs

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1} = \left\{v_{ij}\right\}$$

Vertical Dilution of Precision (VDOP)

$$VDOP \triangleq \frac{\sigma_{z_u}}{\sigma_w} = \sqrt{v_{33}}$$

Horizontal Dilution of Precision (HDOP)

$$HDOP \triangleq \frac{\sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2}}{\sigma_w} = \sqrt{v_{11} + v_{22}}$$

Position Dilution of Precision (PDOP)

$$PDOP \triangleq \frac{\sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2}}{\sigma_w} = \sqrt{v_{11} + v_{22} + v_{33}}$$

Time Dilution of Precision (TDOP)

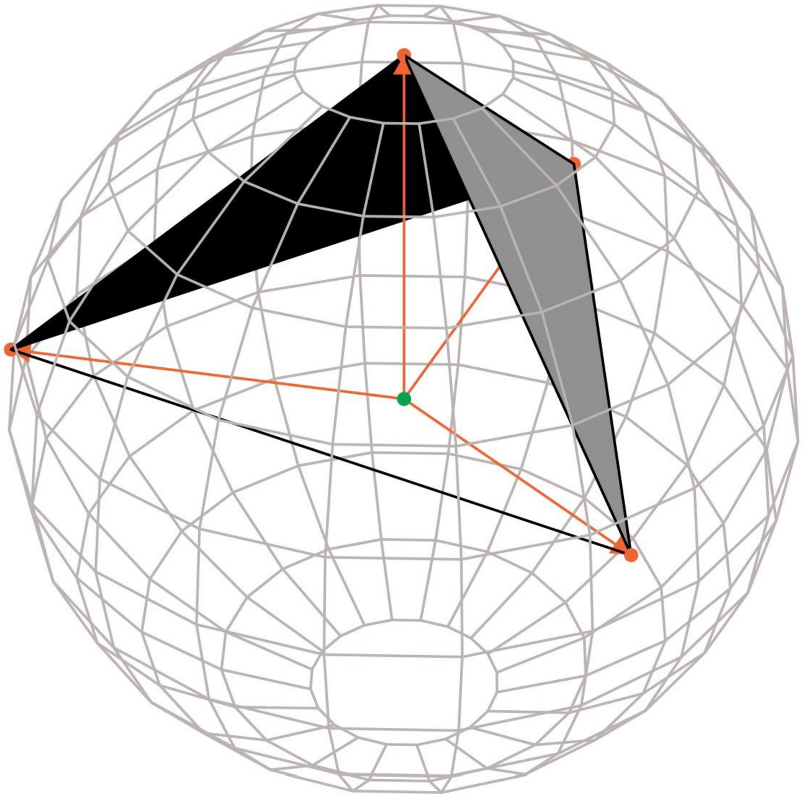
$$TDOP \triangleq \frac{c \cdot \sigma_{\Delta t}}{\sigma_n} = \sqrt{v_{44}}$$

Geometrical Dilution of Precision (GDOP)

$$GDOP \triangleq \frac{\sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2 + c^2 \cdot \sigma_{\Delta t}^2}}{\sigma_w} = \sqrt{v_{11} + v_{22} + v_{33} + v_{44}}$$

Visualizing the DOP

- The four Satellites are the vertices of a *tetrahedron*
- The larger is the *volume* of the tetrahedron, the smaller is the DOP
- What is needed is "diversity" across satellite positions: for instance, If the satellites tend to lie on a plane, the accuracy of the position is very bad
- The DOP values are of the order of the *unity*
- VDOP is larger than HDOP since all satellites are "on the same side" of the receiver vertically



Typical GNSS Pseudorange Error Budget

ERROR SOURCE	*RMS ERROR (m)*
Orbital (Ephemeris)	0.8
Satellite Clock	1
Receiver Noise	0.3
Ionospheric	7
Tropospheric	0.2
Multipath	1
Total UERE	**7.2**

How Much is the DOP ?

H_{min} [°]	Receiver	Number of satellites visible	used	Coefficient HDOP	VDOP	PDOP
0	Leica MX 420	12	9	1.0	1.5	–
	Magnavox MX 200	11	6	1.3	–	–
	SaaB R5 Supreme Nav	12	12	0.7	1.2	1.4
	Simrad MX512	12	8	1.0	2.0	–
5	Furuno GP–33	12	10	–	–	1.76
	Leica MX 420	12	9	1.0	1.4	–
	Magnavox MX 200	11	6	1.3	–	–
	SaaB R5 Supreme Nav	12	12	0.7	1.2	1.4
	Simrad MX512	12	8	1.0	2.0	–
10	Furuno GP–33	12	9	–	–	1.90
	Leica MX 420	11	8	1.0	1.5	–
	Magnavox MX 200	10	6	1.3	–	–
	SaaB R5 Supreme Nav	12	10	0.9	1.7	1.9
	Simrad MX512	12	8	1.0	2.0	–
15	Furuno GP–33	11	8	–	–	2.35
	Leica MX 420	11	8	1.0	1.5	–
	Magnavox MX 200	10	6	1.3	–	–
	SaaB R5 Supreme Nav	12	9	1.0	2.0	2.2
	Simrad MX512	12	7	1.2	2.3	–
20	Furuno GP–33	11	8	–	–	2.36
	Leica MX 420	11	7	1.0	1.6	–
	Magnavox MX 200	8	5	1.5	–	–
	SaaB R5 Supreme Nav	12	9	1.0	2.0	2.2
	Simrad MX512	12	7	1.2	2.3	–
25	Furuno GP–33	11	7	–	–	3.41
	Leica MX 420	11	7	1.0	1.6	–
	Magnavox MX 200	6	5	1.5	–	–
	SaaB R5 Supreme Nav	12	7	1.3	3.5	3.7
	Simrad MX512	12	6	1.4	3.5	–

More than 4 satellites in visibility: Least-Squares solution 1/2

- We can improve on the positioning accuracy using *N*>4 pseudorange measurements coming from more than 4 satellites. The simplest method is just selecting 4 measurements coming from the satellites with the best SNRs – but this does not prevent bad DOP (see the slide before)
- In general, we have *N* observed pseudoranges, and we can write *N* positioning equations; LINEARIZING around a certain ξ_0 we have

$$\mathbf{f}(\xi_0) + \mathbf{A}(\xi - \xi_0) = \boldsymbol{\rho} + \mathbf{w} \quad , \quad \mathbf{A} \triangleq \mathbf{Jf}(\xi_0) \quad \Rightarrow \quad \mathbf{A}\Delta\xi = \Delta\boldsymbol{\rho} + \mathbf{w}$$

where **f** is *N*-dimensional and its Jacobian matrix **A** is *N* x 4

- For the oversized linear set of equations, we can find the *Least-Squares* solution as

$$\Delta\xi_{LS} = \arg\min_{\Delta\xi} \left\| \mathbf{A}\Delta\xi - \Delta\boldsymbol{\rho} \right\|^2$$

More than 4 satellites in visibility: Least-Squares solution 2/2

- From statistical signal processing, the solution to the (linear) problem is found to be

$$\Delta\boldsymbol{\xi}_{LS} = \mathbf{A}_{LS}\Delta\boldsymbol{\rho}$$

where $\mathbf{A}_{LS}$ is the *Least-Squares matrix*

$$\left(\mathbf{A}_{LS}\right)_{4\times N} \triangleq \left(\mathbf{A}^T\mathbf{A}\right)^{-1}_{4\times 4}\mathbf{A}^T_{4\times N}$$

- Coming back to the nonlinear equation, we can solve it recursively by adapting Newton-Raphson's method

$$\boldsymbol{\xi}^{(n+1)} = \boldsymbol{\xi}^{(n)} - \mathbf{A}_{LS}(\boldsymbol{\xi}^{(n)})\left(\mathbf{f}(\boldsymbol{\xi}^{(n)}) - \boldsymbol{\rho}\right)$$

- It is heavy since we need to refresh (recompute) $\mathbf{A}_{LS}$ at each step

DOP of Least-Squares

- Since we now have

$$\Delta\boldsymbol{\xi}_{LS} = \mathbf{A}_{LS}\Delta\boldsymbol{\rho}$$

with

$$\left(\mathbf{A}_{LS}\right)_{4\times N} \triangleq \left(\mathbf{A}^T\mathbf{A}\right)^{-1}_{4\times 4}\mathbf{A}^T_{4\times N}$$

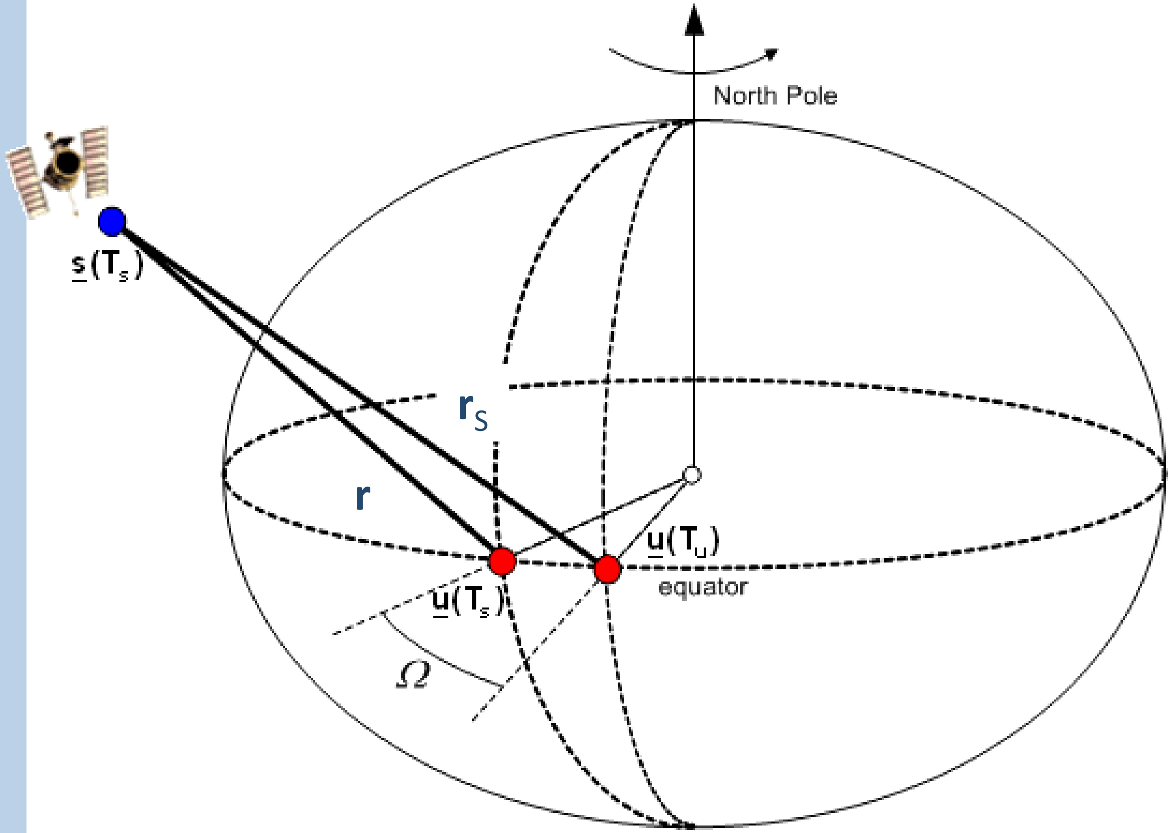
then the new DOP matrix is

$$\left(\mathbf{A}^T_{LS}\mathbf{A}_{LS}\right)^{-1}_{4\times 4}$$

- The «averaging» effect of noise that reduces the DOP wrt the case of 4 satellites only is intrinsic to the computation of the pseudo-inverse

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1}_{4\times 4}$$

The Sagnac effect (1/2)



- The propagation speed of the em wave generated by the satellite does *not* depend on the speed of the satellite (special relativity) IN AN INERTIAL FRAME
- During the propagation time of the satellite signal (about 67 ms), a clock on the surface of the Earth will experience a time-shift with respect to a resting (inertial) reference frame
- The measured range is $\mathbf{r}_S$ rather than **r**.
- It is also called “Earth Rotation Correction”

The Sagnac effect (2/2)

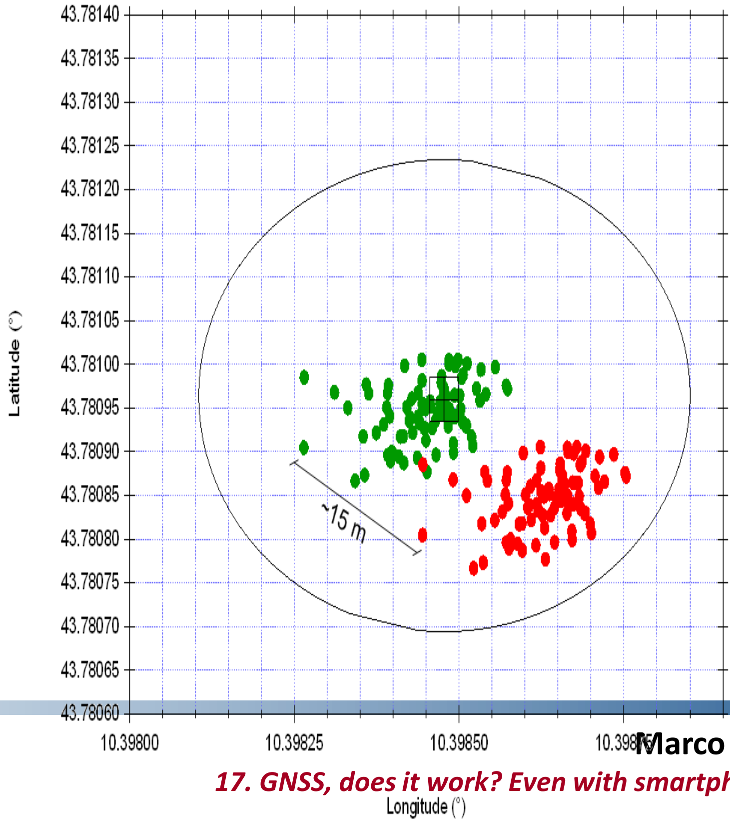
(Non-inertial) ECEF ⇔ (Inertial) ECI

$$r \cong \sqrt{r_S^2 - 2\Omega\left[x_s \cdot y_u - y_s \cdot x_u\right]} = \sqrt{r_S^2 - 2\omega_E \frac{r_S}{c}\left[x_s \cdot y_u - y_s \cdot x_u\right]}$$

$\omega_E = 7.292115 \times 10^{-5}$ rad/s (WGS-84)

$c = 2.99792458 \times 10^{8}$ m/s (WGS-84)

$$\left|r - r_S\right| \cong 3 \div 10 \text{ m} \quad \textit{for each satellite}$$



Dip. Ingegneria dell'Informazione
University of Pisa, Italy

International Time Scales

- **Any GNSS receiver can "lock" to the satellites' clock, since it derives its own clock bias Δt. Where does the satellites' time come from? Who determines and keep it ? What is its relation with time references available on the Internet?**

- **Universal Time (UT1)**
  - Aka astronomical time or solar time, is determined by the position of the Sun relative to the observer. The exact duration of a UT1 day is not always the same - UT1 does not flow uniformly
- **International Atomic Time (TAI)**
  - Metrologic timescale maintained by the *Bureau des Poids et Measure* (BIPM)
  - TAI is defined as a coordinate timescale in a geocentric reference frame with the SI second as the scale unit realized on the rotating geoid
- **Coordinated Universal Time (UTC)**
  - Stepped atomic time scale based on the rate of TAI adjusted by the addition or deletion of integer seconds, known as leap seconds, to maintain the time within ±0.9 s of Universal Time (UT1),

GNSS Time(s) vs. UTC

UTC – GPST	$0\,h - n + 19\,s + C_0$	GPS Time (GPST) is steered to UTC(USNO), C_0 is required to be less than 1 μs but is typically less than 20 ns
UTC – GLST	$-3\,h + 0\,s + C_1$	GLONASST (GLONASS Time) is steered to UTC(SU) including leap seconds. C_1 is required to be less than 1 ms. Note that GLONASST is offset from UTC by –3 hours corresponding to the offset of Moscow local time from the Greenwich meridian.
UTC – GST	$0\,h - n + 19\,s + C_2$	Galileo Time (GST) is steered to a set of European Union UTC(k) realization and C_2 is nominally less than 50 ns.
UTC – BDT	$0\,h - n + 33\,s + C_3$	BeiDou Time (BDT) is steered to UTC(NTSC) and C_3 is specified to be maintained less than 100 ns.

- Each GNSS has its own reference time – there are *offsets* between different GNSSs
- Each offset is separated into an integer number of seconds and its fractional (subsecond) component C_i.
- n = TAI –UTC denotes the integer second offset between International Atomic Time and Coordinated Universal Time (e.g., n = 36 s starting on 1 July 2015)